# Real-time dynamics of quantum tunneling motivated by Lefschetz-thimble path integral

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# Motivation

Direct semiclassical description of quantum tunneling from the viewpoint of real-time path integral,

$$\int \mathcal{D}x(t) \exp(\mathrm{i}S[x(t)]/\hbar).$$

Problem: Classical eom $\delta S = 0$ does not have tunneling solutions! Naively, semiclassical description seems to be impossible.

# Simple example: Airy integral

Let's consider a one-dimensional oscillatory integration:

$$\mathrm{Ai}(a) = \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi} \exp \mathrm{i} \left( \frac{x^3}{3} + ax \right).$$
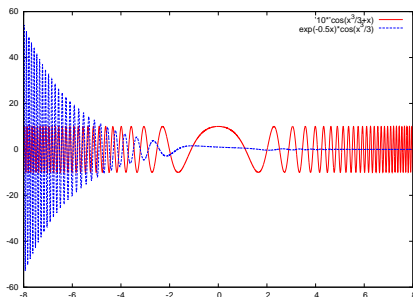
What is the MFA of this system?



Figure: Real parts of integrands for $a = 1$ ($\times 10$) & $a = 0.5\mathrm{i}$

# New technology on path integral

Complex paths open a new way to compute path integrals.

It's better to circumvent the sign problem in order for a semiclassical method.

$\Rightarrow$ Lefschetz-thimble path integral (E.Witten, arXiv:1001.2933, arXiv:1009.6032)

$$\int \mathcal{D}x \exp iS[x]/\hbar = \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} \mathcal{D}z \exp iS[z]/\hbar.$$

# Find complex classical solutions

Idea:

1. Find complex classical solutions $z_\sigma$ of eom $\delta S = 0$.
2. Consider appropriate integration contours $\mathcal{J}_\sigma$ around $z_\sigma$.
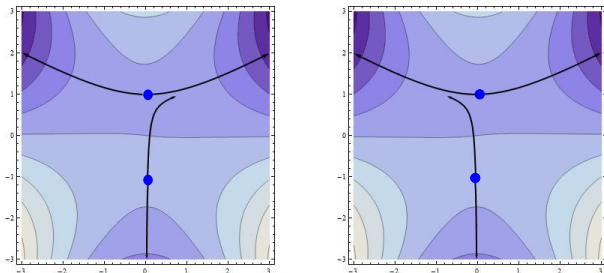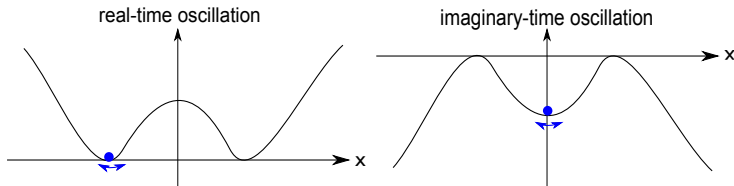3. Path integral on $\mathcal{J}_\sigma$ can be computed without the sign problem.



Figure: Lefschetz thimbles for the Airy integral $a = \exp \pm i0.1$.

# Double-well potential

Complex-energy conservation: $\left(\dfrac{\mathrm{d}z}{\mathrm{d}t}\right)^2 + (z^2 - 1)^2 = p^2$

Two different origins of oscillations
$\Rightarrow$ Solutions show double-periodicity!



real-time oscillation

imaginary-time oscillation

## Double-well potential

The list of parameters $p$ can be obtained as ($n, m \in \mathbb{Z}$)

$$n \frac{K(\sqrt{(p+1)/2p})}{\sqrt{2p}} + m \frac{\mathrm{i}K(\sqrt{(p-1)/2p})}{\sqrt{2p}} = \frac{t_f - t_i}{2} + (\text{bdry. terms}).$$
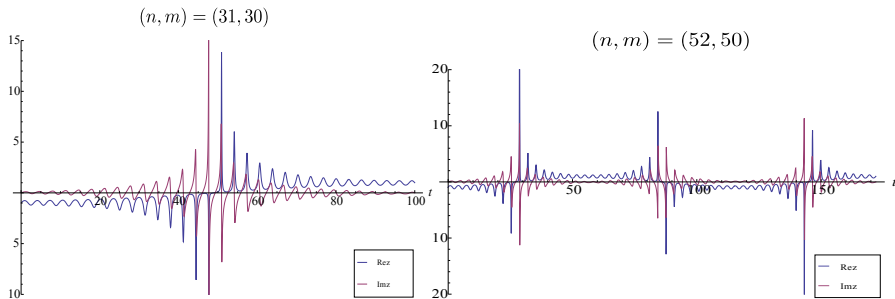
Short-time asymptotic behaviors of the classical action:

$$\mathcal{I}[z_{(n,m)}] \simeq \mathrm{i} \frac{2K(1/\sqrt{2})^4}{3} \frac{(n + \mathrm{i}m)^4}{(t_f - t_i)^3}.$$

# Complex solutions for quantum tunneling

In order for the one-instanton action $iS[z] \simeq -S_{\text{inst.}} = -4/3$, complex solutions must be (highly-)oscillatory in the complexified space, i.e., $n, m \simeq O(t_f - t_i)$
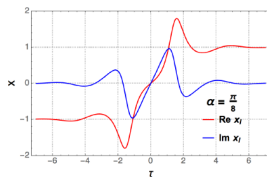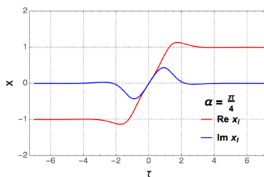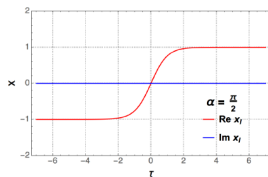(Me & Koike, arXiv:1406.2386):

# Connection to the instanton calculus

This solution has a close relationship with one-instanton
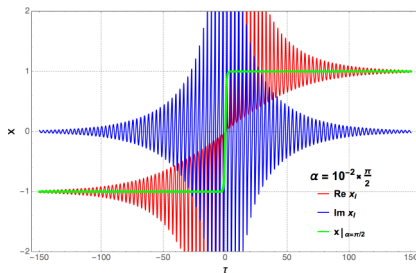(Cherman & Ünsal, arXiv:1408.0012):

$$x_\alpha(\tau) = \tanh[(\tau - \tau_0)e^{-i(\alpha - \pi/2)}]$$

# Connection to the instanton calculus

The action of this solution is that of the one-instanton solution at **any** $\alpha$ (Cherman & Ünsal, arXiv:1408.0012):

$$S_\alpha = \frac{e^{-i\alpha}}{2\hbar} \int d\tau [e^{2i\alpha}(\partial_\tau x_\alpha)^2 - (x_\alpha^2 - 1)^2] = \frac{4}{3}$$



$\alpha = 10^{-2} \times \frac{\pi}{2}$
— Re $x_l$
— Im $x_l$
— $x|_{\alpha=\pi/2}$

# Summary & Perspectives

## Summary

- Real-time dynamics will become calculable in a nonperturbative way with Lefschetz-thimble path integrals.

- Exact semi-classical description of quantum tunneling is considered.

## Perspectives

- Application to real-time dynamics of topological objects in field theories.

- Sign problem, Resurgence trans-series, ....